# DISCRETE SCALE INVARIANCE IN HOLOGRAPHY AND AN ARGUMENT AGAINST THE COMPLEXITY = ACTION PROPOSAL*

Mario Flory

Institute of Theoretical Physics, Jagiellonian University
Łojasiewicza 11, 30-348 Kraków, Poland
mflory@th.if.uj.edu.pl

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The AdS/CFT correspondence often motivates research on questions in gravitational physics whose relevance might not be immediately clear from a purely GR perspective, but which are nevertheless interesting. In these proceedings, we summarise two such results recently obtained by the author. One concerns, broadly speaking, the possible isometry groups of a spacetime sourced by physical matter. The other one provides a possible argument against the recently proposed complexity = action conjecture.

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## 1. Introduction

The AdS/CFT correspondence allows to view gravitational physics in terms of a dual quantum field theory interpretation. This often motivates research on questions in gravitational physics whose relevance might not be immediately clear from a purely General Relativity (GR) perspective, but which are nevertheless interesting. As an illustration of this claim, let us look at the complete GR action in an arbitrary spacetime region $\mathcal{W}$, following the summary of [1] (see also references therein)

$$\mathcal{A} \propto \frac{1}{2}\int\limits_{\mathcal{W}} (R-2\Lambda)\sqrt{-g}\,\mathrm{d}^3x + \sum_{\mathcal{T}_i}\int\limits_{\mathcal{T}_i} K\sqrt{-\gamma}\,\mathrm{d}^2x + \sum_{\mathcal{S}_i}\int\limits_{\mathcal{S}_i} K\sqrt{\gamma}\,\mathrm{d}^2x$$
$$+\sum_{\mathcal{N}_i}\int\limits_{\mathcal{N}_i} \kappa\,\mathrm{d}\lambda\sqrt{\rho}\,\mathrm{d}x + \sum_{\mathcal{J}_i}\int\limits_{\mathcal{J}_i} \eta_{\mathcal{J}_i}\sqrt{\rho}\,\mathrm{d}x + \sum_{\mathcal{N}_i}\int\limits_{\mathcal{N}_i} \theta\log(|\theta\ell_c|)\mathrm{d}\lambda\sqrt{\rho}\,\mathrm{d}x\,.$$

Herein, only the first term dates back to Einstein and Hilbert, the second and third are the necessary boundary terms on spacelike and timelike boundaries

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$\mathcal{T}_i, \mathcal{S}_i$ of $\mathcal{W}$ derived in the 70s, while the remaining terms were studied in more modern times, and describe the roles of joints $\mathcal{J}_i$, null boundaries $\mathcal{N}_i$, and so-called counter terms (the last term) giving the action $\mathcal{A}$ a well-defined reparametrization-invariant value [1]. Hence, despite the centennial of General Relativity, a complete picture of the gravitational action $\mathcal{A}$ was only recently formulated, and this work was strongly motivated by AdS/CFT, specifically the recent complexity = action (CA) conjecture [2], to which we will return in Sec. 3.

## 2. Discrete scale invariance in holography

Consider the question whether for a given (Lie)group there exists a smooth (semi)-Riemannian metric with that group as its isometry group, which in Einsteins equations can be sourced by matter satisfying the weak energy condition (and possibly also satisfies additional requirements). This seems like a question, if maybe a bit obscure, belonging squarely into the realm of GR and differential geometry. However, in holography it takes on a life of its own: A well-known corner-stone of the AdS/CFT correspondence is that the isometry group of $\mathrm{AdS}_{d+1}$ corresponds to the conformal group of a $\mathrm{CFT}_d$. As argued in [3], breaking the isometry group of AdS down to the Poincaré group combined with a *discrete scale invariance* (DSI) would correspond to the holographic description of a cyclic RG flow — a highly unusual and potentially very interesting phenomenon. In fact, two models seeming to do just that were presented in [3], one "bottom-up" and one "top-down". These models gave rise to solutions to Einstein's equations of the form of

$$\mathrm{d}s^2 = \mathrm{e}^{2C(w,\theta)}\left(\mathrm{e}^{2w/L}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + \mathrm{d}w^2\right) + \mathrm{e}^{2B(w,\theta)}\left(\mathrm{d}\theta + A(w,\theta)\mathrm{d}w\right)^2$$

with specific periodic functions $A, B$ and $C$, such that only Poincaré invariance in the boundary directions $t, \vec{x}$ combined with DSI is manifestly preserved.

However, as pointed out in [4], the only way to *prove* that the AdS-isometry group was successfully broken down to a subgroup is to analytically find all linearly independent solutions of the Killing equations

$$\nabla_\mu \mathcal{K}_\nu + \nabla_\nu \mathcal{K}_\mu = 0$$

in the metric above, and checking the isometry algebra formed by the Lie-brackets of these Killing vector fields. This was done in [4] proving that the bottom-up model of [3] still exhibits a full $\mathrm{AdS}_d$-like isometry group, and hence does not describe genuine DSI. However, the top-down model of [3] does seem to describe genuine DSI, and hence deserves further study.

## 3. An argument against the complexity = action proposal

Let us now return to the Einstein action written down in Sec. 1, specifically the counter term. As pointed out in [5] using Raychaudhuri's equation, in a $2+1$-dimensional vacuum spacetime, this term is a total derivative and can be rewritten as

$$\mathcal{A}_{\text{counter}} = \int \int_{\lambda_{\min}}^{\lambda_{\max}} (\partial_\lambda \sqrt{\rho}) \log(|\theta \ell_c|) \mathrm{d}\lambda \mathrm{d}x = \int \left[\sqrt{\rho} \log(|\theta \ell_c'|)\right] \Big|_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}x \,, \quad (1)$$

where $\lambda$ is a coordinate along the null-rays foliating the null-boundary $\mathcal{N}_i$, $x$ is a coordinate enumerating these null-rays, $\sqrt{\rho}$ is a volume-element along spacelike curves parametrized by $x$, and the expansion is defined as $\theta = \frac{1}{\sqrt{\rho}} \partial_\lambda \sqrt{\rho}$. Thus, the contribution from this term can be written as an integral along the joint-curves where null-boundaries of $\mathcal{W}$ collide with other boundaries, or each other. This has an important consequence [5]: When applying an infinitesimal *local conformal transformation* (with infinitesimal expansion parameter $\sigma \ll 1$) to the simple Wheeler–DeWitt patch in $\mathrm{AdS}_3$ which corresponds to the vacuum state of the dual theory, the leading behaviour of the change of the action will come from this term and read

$$\mathcal{A}_{\text{counter}} \sim \sigma \log \sigma \,. \quad (2)$$

This result has immediate consequences for the CA proposal of [2], which claims that the value of the gravity action $\mathcal{A}$, evaluated on a Wheeler–DeWitt patch $\mathcal{W}$, is holographically equivalent to a measure of complexity $\mathcal{C}(\psi)$ of the dual field theory state $\psi$. Complexity herein is understood as a distance measure on the space of states, and hence is subject to consistency requirements such as positivity, the triangle inequality, *etc.* [6][1].

However, we can now show a contradiction between the following three assumptions (for $\sigma \ll 1$):

1. The infinitesimal local conformal transformation is generated by a unitary operator $U(\sigma) = \mathbb{1} + \sigma V + \mathcal{O}(\sigma^2)$ ($V$ can be explicitly expressed in terms of Virasoro generators) with complexity $\mathcal{C}(U(\sigma)) = \sigma \mathcal{K}' + \mathcal{O}(\sigma^2)$, $0 < \mathcal{K}' < +\infty$. The latter condition follows from the positive homogeneity property of Nielsen's proposal [6] for complexity with the additional assumption of using finite "penalty factors".

[1] Technically, as in [5], we assume that complexity is primarily defined as a distance measure on the space of unitariy operators, which after a choice of reference state induces a distance measure on the space of states that can be reached from the reference state by unitary transformations.

2. The change of complexity of the dual state $\psi$ caused by applying the operator $U$ has to be less than the complexity of $U$: $\mathcal{C}(U(\sigma)) \geq |\delta\mathcal{C}(\psi)|$ (triangle inequality).

3. $|\delta\mathcal{C}(\psi)| = \mathcal{K}|\sigma\log(\sigma)|$, $0 < \mathcal{K} < +\infty$, (equation (2)) due to the CA proposal when using the action as given in Sec. 1.

Together, these three assumptions would imply that $\sigma\mathcal{K}' \geq \mathcal{K}|\sigma\log(\sigma)|$ as $\sigma \to 0$ for positive finite constants $\mathcal{K}$ and $\mathcal{K}'$, which is false. Hence, any definition of field-theory complexity that satisfies assumptions 1 and 2 cannot be exactly dual to the CA proposal in $\mathrm{AdS}_3/\mathrm{CFT}_2$ with the counter-terms chosen as in Sec. 1 (which implies assumption 3 by equation (2) as shown in [5]).

In fact, a similar argument can be applied to the complexity change caused by time evolution by an infinitesimal time-step $\delta t$ beyond the critical time $t_{\mathrm{c}}$ in [7]. This would lead similarly to a complexity change of the form of $|\delta\mathcal{C}(\psi)| = \mathcal{K}|\delta t\log(\delta t)|$, and the same contradiction as above could be constructed. Interestingly, in this case, the contradiction even arises in general dimensions and independently of whether the counter term is added to the gravity action or not, however, it was also argued in [7] that this problem can be solved by requiring the complexity as a function of time to be smoothed out over a certain time scale.

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